Written Exam at the Department of Economics winter 2019-20

Financial Econometrics A

Final Exam

January 14, 2020

(3-hour closed book exam)

Answers only in English.

This exam question consists of 5 pages in total

Falling ill during the exam

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- contact an invigilator who will show you how to register and submit a blank exam paper.
- leave the examination.
- contact your GP and submit a medical report to the Faculty of Social Sciences no later than five (5) days from the date of the exam.

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You cheat at an exam, if during the exam, you:

- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

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Please note there is a total of **8** questions that you should provide answers to. That is, **3** questions under *Question A*, and **5** under *Question B*.

Question A:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = \sigma_t z_t, \quad \sigma_t^2 = \omega + \alpha x_{t-1}^2$$

with (z_t) an *i.i.d.* process, and z_t is scaled t-distributed and given by

$$z_t = \sqrt{\frac{v-2}{v}} \tau_t,$$

where τ_t is t-distributed with $v = 5$ degrees of freedom. In particular, we have

$$E\tau_t = 0,\ V(\tau_t) = E\left(\tau_t^2\right) = \frac{v}{v-2} = \frac{5}{3} \text{ and } E\left(\tau_t^4\right) = \frac{3v^2}{(v-4)(v-2)} = 25.$$

Moreover, the model parameters satify that $\omega > 0$ and $\alpha \geq 0$.

Question A.1: Show that x_t is weakly mixing with $Ex_t^4 < \infty$ if $\alpha < 1/3$.

Question A.2: Consider the extended ARCH(2) model,

$$x_t = \sigma_t z_t, \quad \sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta x_{t-2}^2,$$

with $\omega > 0$ and $\alpha, \beta \geq 0$, and z_t still i.i.d. scaled t-distributed with $v = 5$. The Gaussian QMLE $\hat{\theta} = \left(\hat{\omega}, \hat{\alpha}, \hat{\beta}\right)'$ is obtained by maximizing $L(\theta)$ given by

$$L(\theta) = -\frac{1}{T} \sum_{t=1}^{T} \left(\log \sigma_t^2(\theta) + \frac{x_t^2}{\sigma_t^2(\theta)} \right),$$
$$\sigma_t^2(\theta) = \omega + \alpha x_{t-1}^2 + \beta x_{t-2}^2.$$

It follows that,

$$S_\beta(\theta) = \partial L(\theta)/\partial\beta = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{x_t^2}{\sigma_t^2(\theta)} - 1\right)\frac{x_{t-2}^2}{\sigma_t^2(\theta)}.$$

With $\theta_0 = (\omega_0, \alpha_0, 0)$ (that is, the true value of β is zero), show that if $0 < \alpha_0 < 1/3$ then

$$\sqrt{T}S_\beta(\theta_0) \xrightarrow{d} N(0, \Omega)$$

with

$$\Omega = 8E\left(\frac{x_{t-2}^2}{\omega_0 + \alpha_0 x_{t-1}^2}\right)^2.$$

Question A.3:

In the ARCH(2) model we wish to test the hypothesis $\beta = 0$ by the Quasi likelihood ratio statistic

$$QLR = \left(L(\hat{\theta}) - L\left(\tilde{\theta}\right)\right),$$

where $\tilde{\theta}$ maximizes $L(\theta)$ with $\beta = 0$. The asymptotic distribution of the QLR statistic is not χ_1^2. Explain why it is not χ^2. And discuss which distribution it has. Relate your answer to Question A.2.

Question B:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = b_{s_t} + a_{s_t} x_{t-1} + \varepsilon_t, \tag{B.1}$$

where the error term ε_t satisfies

$$\varepsilon_t \sim i.i.d.N(0, \sigma^2). \tag{B.2}$$

Moreover,

$$b_{s_t} = 1(s_t = 1)b_1 + 1(s_t = 2)b_2, \tag{B.3}$$

and

$$a_{s_t} = 1(s_t = 1)a_1 + 1(s_t = 2)a_2, \tag{B.4}$$

where s_t is a state variable that takes values in $\{1, 2\}$ according to the transition probabilities

$$P(s_t = j | s_{t-1} = i) = p_{ij}, \tag{B.5}$$

and $1(s_t = i) = 1$ if $s_t = i$ and $1(s_t = i) = 0$ if $s_t \neq i$ for $i = 1, 2$. We assume throughout that the processes (ε_t) and (s_t) are independent. The model parameters satisfy $b_1, b_2, a_1, a_2 \in \mathbb{R}$ and $\sigma^2 > 0$.

Question B.1: When is the process (s_t) weakly mixing?

Question B.2: Let $f(x_t|x_{t-1}, s_t)$ denote the conditional density of x_t given (x_{t-1}, s_t). Provide an expression for $f(x_t|x_{t-1}, s_t)$.

Question B.3: In the following we assume that $p_{11} = 1 - p_{22} =: p \in (0, 1)$ such that (s_t) is an *i.i.d.* process with $P(s_t = 1) = p$. This implies that $f(x_t|x_{t-1}) = f(x_t|x_{t-1}, s_t = 1)P(s_t = 1) + f(x_t|x_{t-1}, s_t = 2)P(s_t = 2) > 0$. Use this to show that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$ when

$$a_1^2 p + a_2^2 (1 - p) < 1.$$

Question B.4: Consider the *restricted* version of the model in (B.1)-(B.4), where $b_1 = b_2 = 0$. Maintaining the assumptions from Question B.3, and assuming that (s_t) is *observed*, we consider the log-likelihood function (up to a constant),

$$L_T(\theta) = \sum_{t=1}^{T} \Bigg[1(s_t = 1) \left\{ -\frac{1}{2}\log(\sigma^2) - \frac{(x_t - a_1 x_{t-1})^2}{2\sigma^2} + \log(p) \right\}$$
$$+ 1(s_t = 2) \left\{ -\frac{1}{2}\log(\sigma^2) - \frac{(x_t - a_2 x_{t-1})^2}{2\sigma^2} + \log(1-p) \right\} \Bigg],$$

where $\theta = (p, a_1, a_2, \sigma^2)$. Show that the Maximum Likelihood Estimator for a_1 is

$$\hat{a}_1 = \frac{\sum_{t=1}^{T} 1(s_t = 1) x_t x_{t-1}}{\sum_{t=1}^{T} 1(s_t = 1) x_{t-1}^2}.$$

Assume that the joint process (s_t, x_{t-1}) is weakly mixing such that $E[x_{t-1}^2] < \infty$. Argue that $\hat{a}_1 \xrightarrow{p} a_1$ as $T \to \infty$.

Question B.5: Suppose that the process (s_t) is *unobserved*, but still satisfies the *i.i.d.* assumption, i.e. $p_{11} = (1 - p_{22}) = p \in (0, 1)$. Then the estimator $\hat{a}_1$ derived in Question B.4 is infeasible. Consider instead the function

$$L_T^{\dagger}(\theta) = E[L_T(\theta) | x_1, ... x_T].$$

It holds (still with $b_1 = b_2 = 0$) that

$$L_T^{\dagger}(\theta) = \sum_{t=1}^{T} \Bigg[P_t^{\dagger}(1) \left\{ -\frac{1}{2}\log(\sigma^2) - \frac{(x_t - a_1 x_{t-1})^2}{2\sigma^2} + \log(p) \right\}$$
$$+ (1 - P_t^{\dagger}(1)) \left\{ -\frac{1}{2}\log(\sigma^2) - \frac{(x_t - a_2 x_{t-1})^2}{2\sigma^2} + \log(1-p) \right\} \Bigg].$$

where $P_t^{\dagger}(1) = P(s_t = 1 | x_t)$.
Explain briefly the role of $L_T^{\dagger}(\theta)$ for the estimation of θ.